The auxiliary region method for coupling PDE and Brownian-based dynamics for reaction-diffusion systems

Cameron Smith (Cohort 3)

SAMBa Conference 2019
08/07/19

Outline

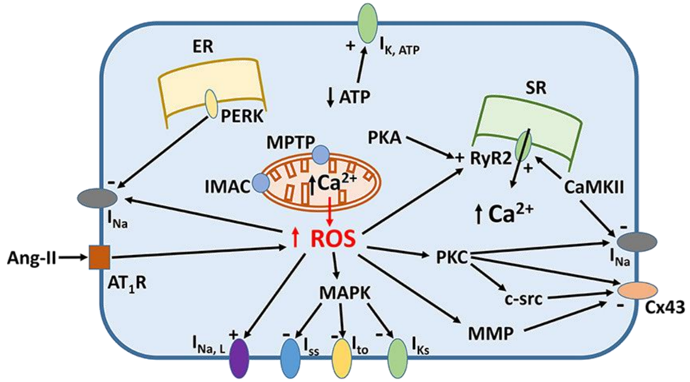
Reaction-diffusion systems

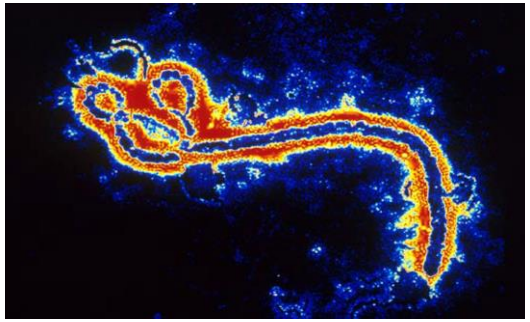
Spatially extended hybrid methods

The auxiliary region method (ARM)

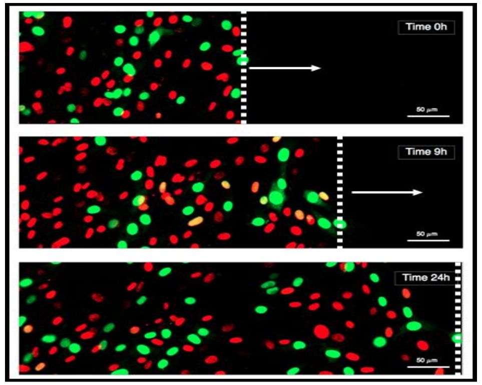
Reaction-diffusion systems

SAMBa

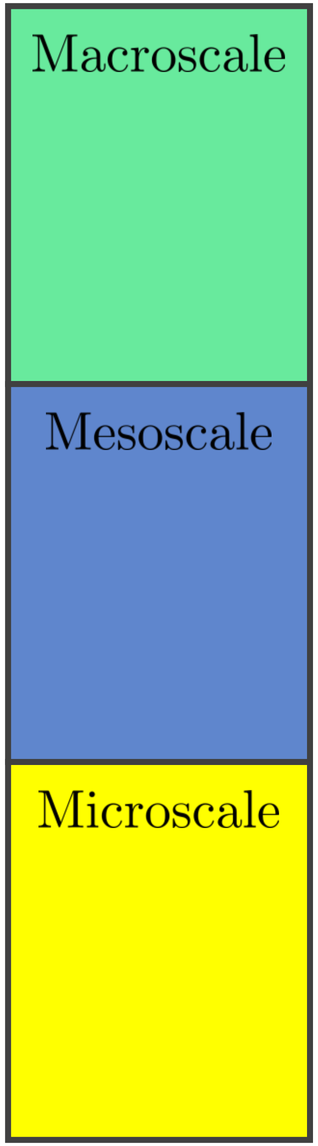




Reaction-diffusion



Modelling



Modelling

Macroscale

Mesoscale

Microscale

Partial differential equation

$$\frac{\partial u}{\partial t} = D\nabla^2 u + \mathcal{R}[u]$$

Diffusion

Reactions

- ✓ Many numerical techniques
- ✓ Amenable to analysis
- ✓ Fast to simulate
- ✗ Unsuitable for low copy numbers
- ✗ Can be inaccurate

Modelling

Macroscale

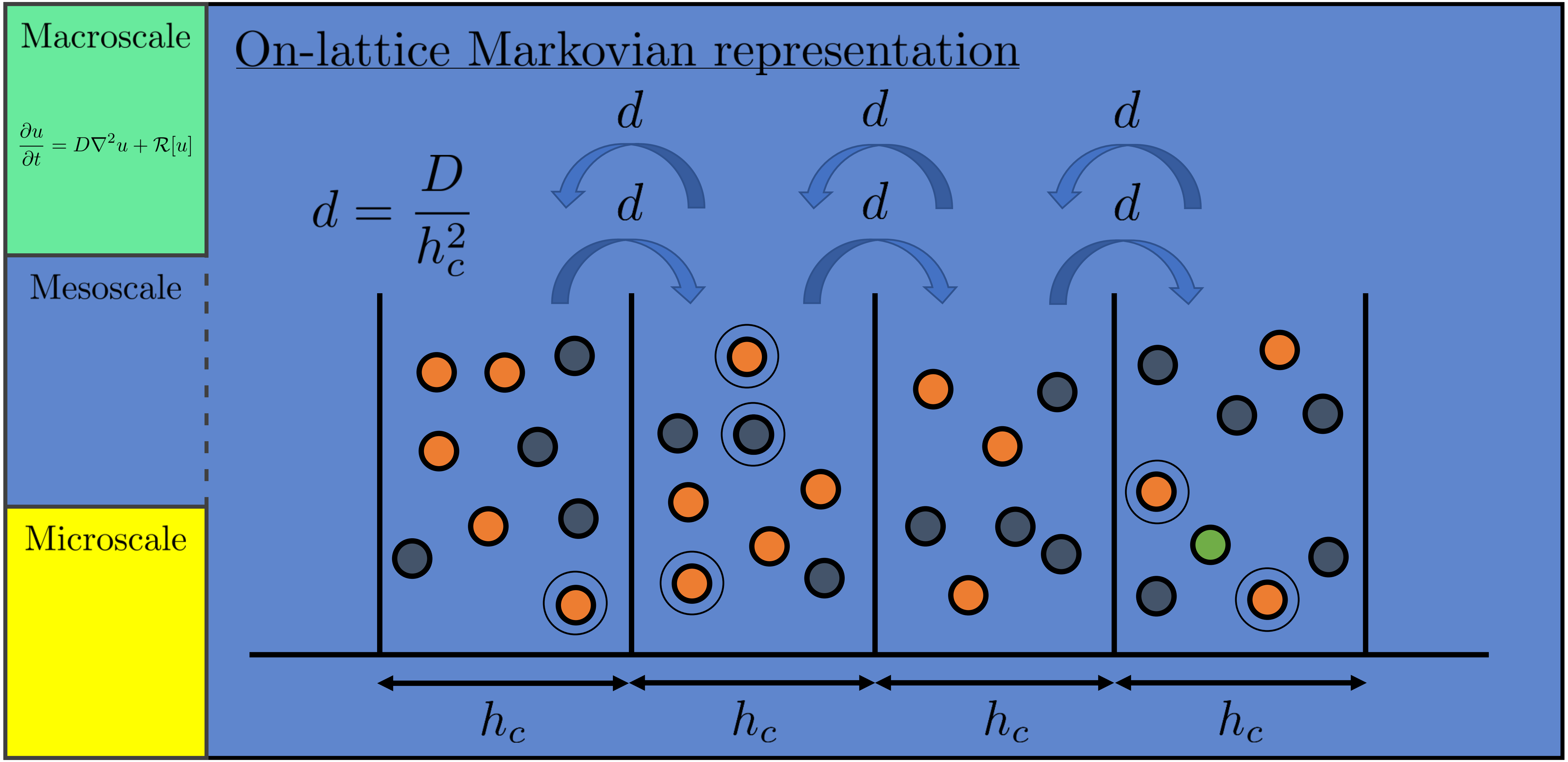
$$\frac{\partial u}{\partial t} = D\nabla^2 u + \mathcal{R}[u]$$

Mesoscale

Microscale

On-lattice Markovian representation

Modelling


Macroscale
$\frac{\partial u}{\partial t} = D\nabla^2 u + \mathcal{R}[u]$
Mesoscale
Microscale
On-lattice Markovian representation
$d = \frac{D}{h_c^2}$
d
d
d
d
d
d
h_c
h_c
h_c
h_c

Modelling

Macroscale

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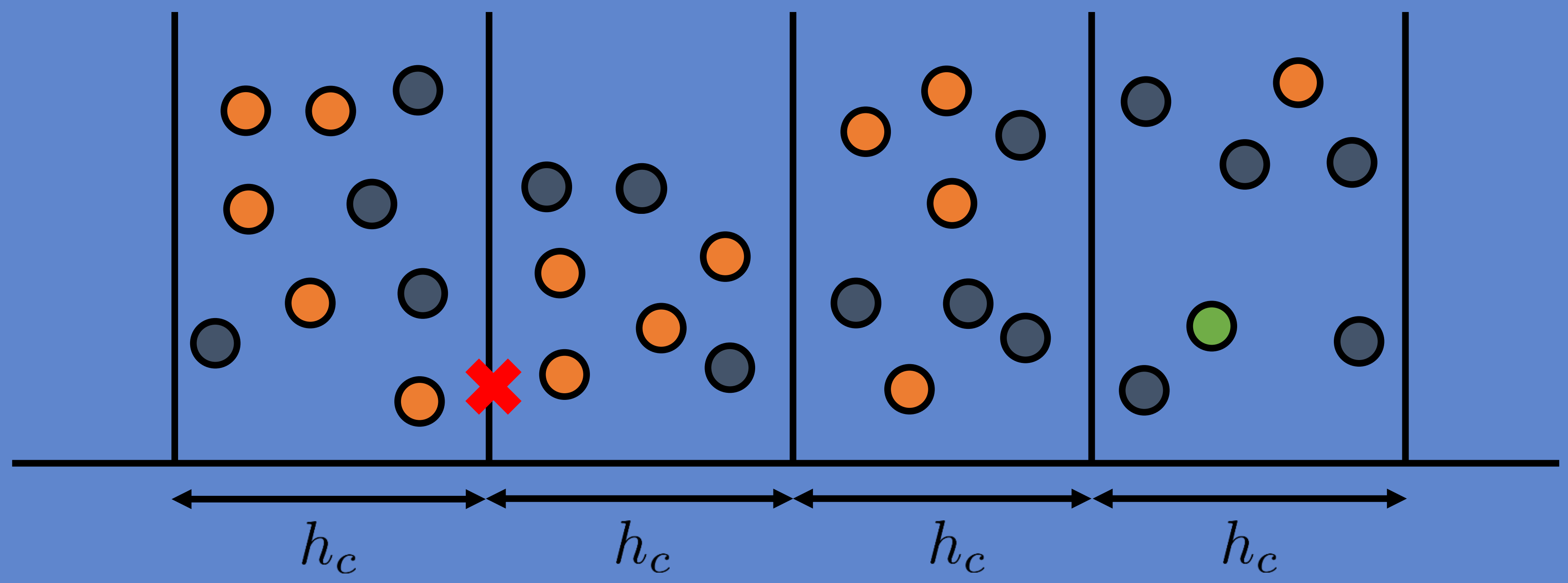
Mesoscale

Microscale

On-lattice Markovian representation



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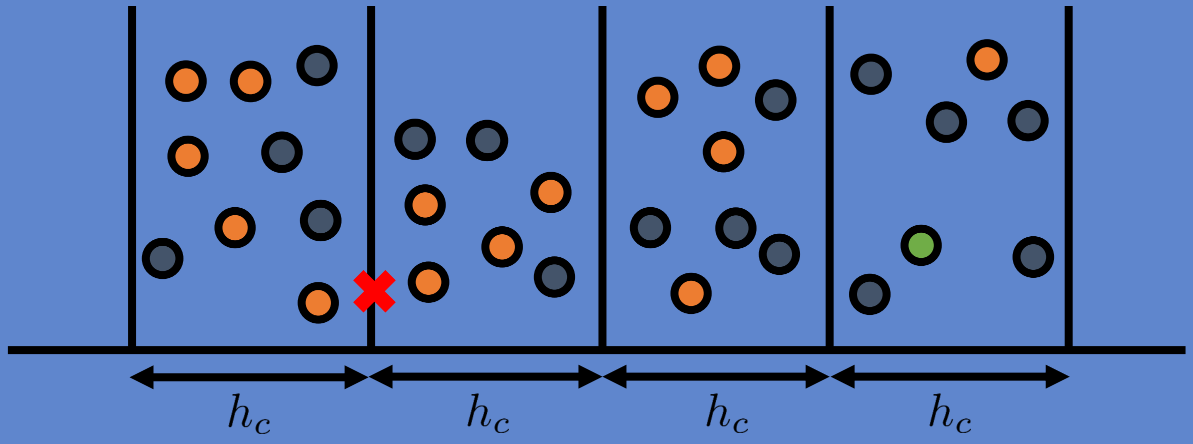


- ✓ Many numerical techniques
- ✓ Stochastic information
- ✓ Reactions are simple to implement

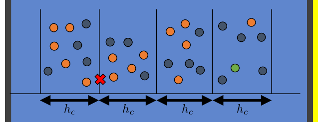
- ✗ Slow for large numbers of particles
- ✗ Lose particle “locations”

Modelling

Macroscale

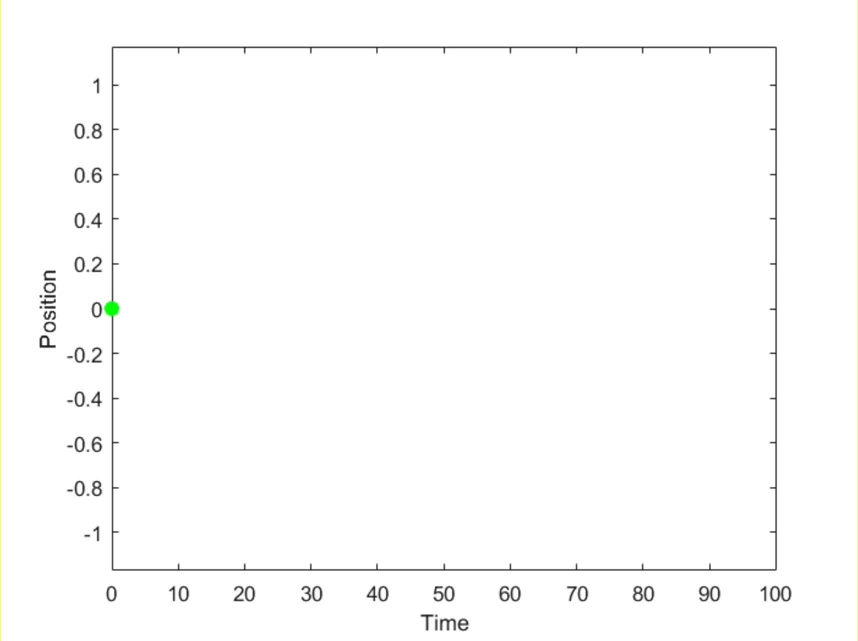
$$\frac{\partial u}{\partial t} = D\nabla^2 u + \mathcal{R}[u]$$

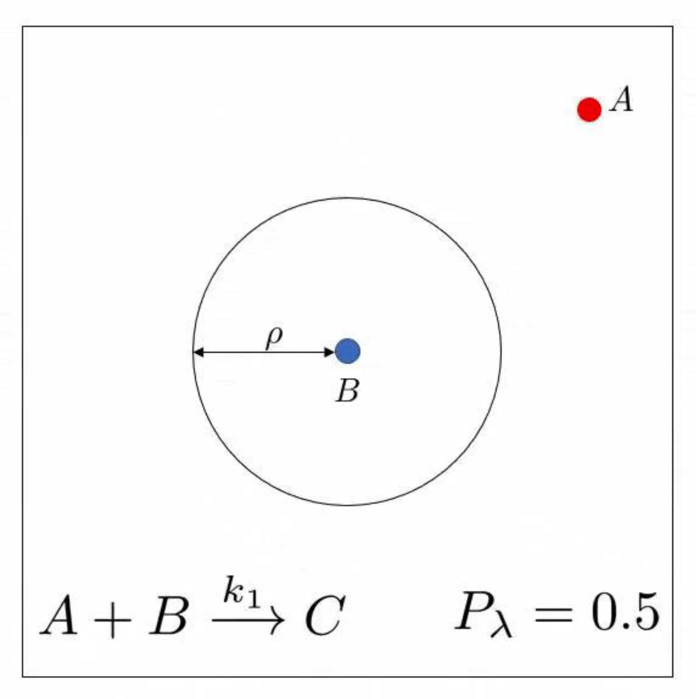
Mesoscale



Microscale

Off-lattice Brownian-based





Modelling

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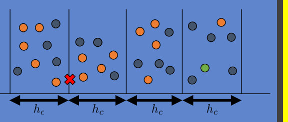
Mesoscale



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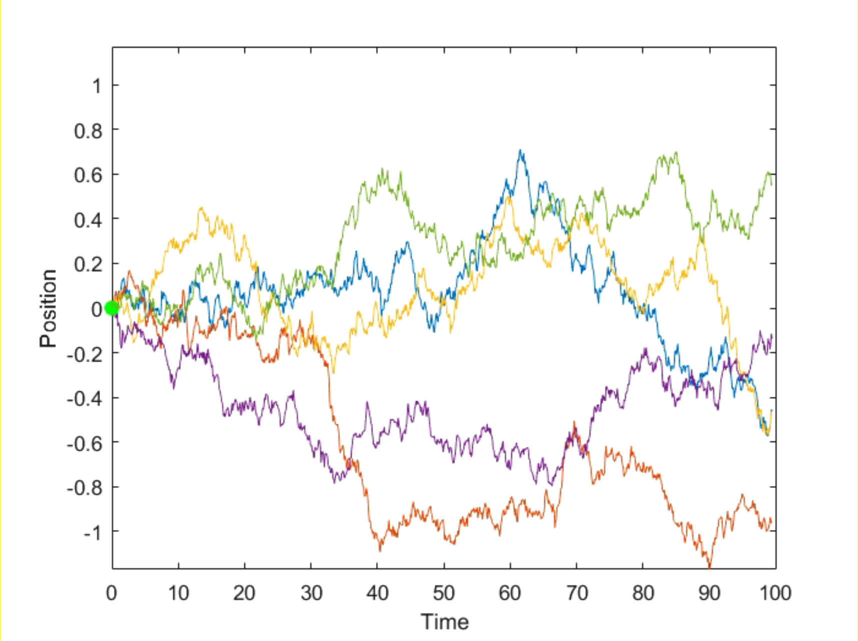

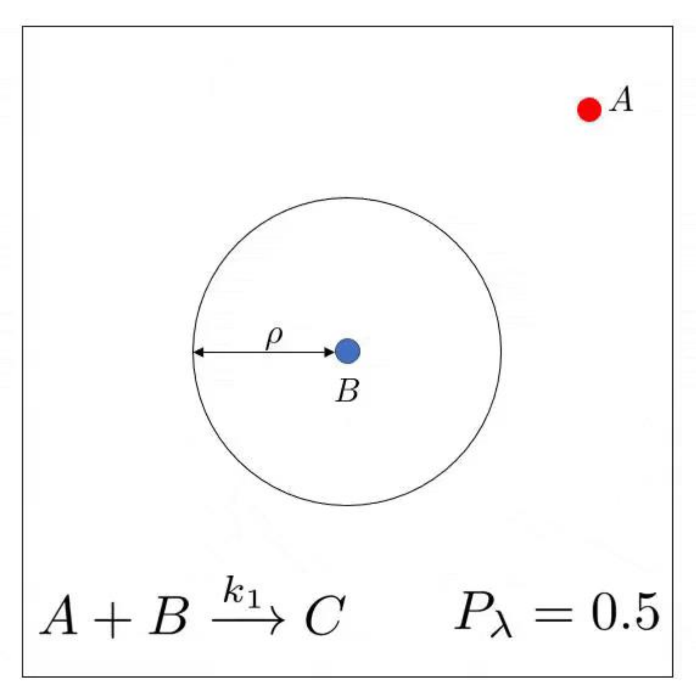

Modelling

Macroscale

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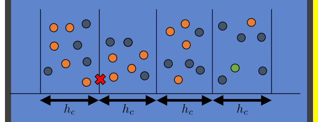
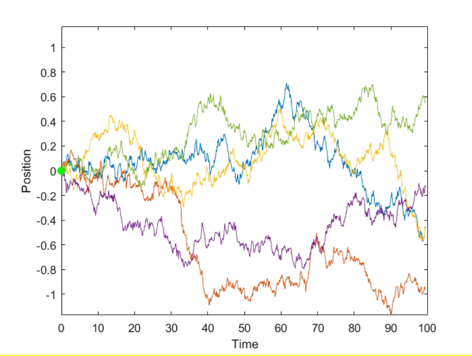
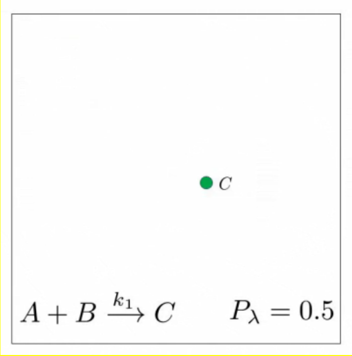
Mesoscale



Microscale

Off-lattice Brownian-based





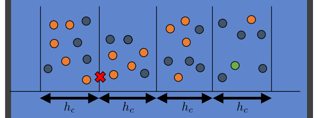
- ✓ Individual level behaviour
- ✓ Stochastic information
- ✗ Slow to implement
- ✗ May require pairwise distances

Modelling

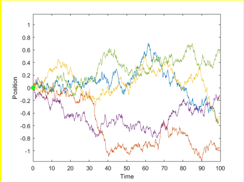
Macroscale

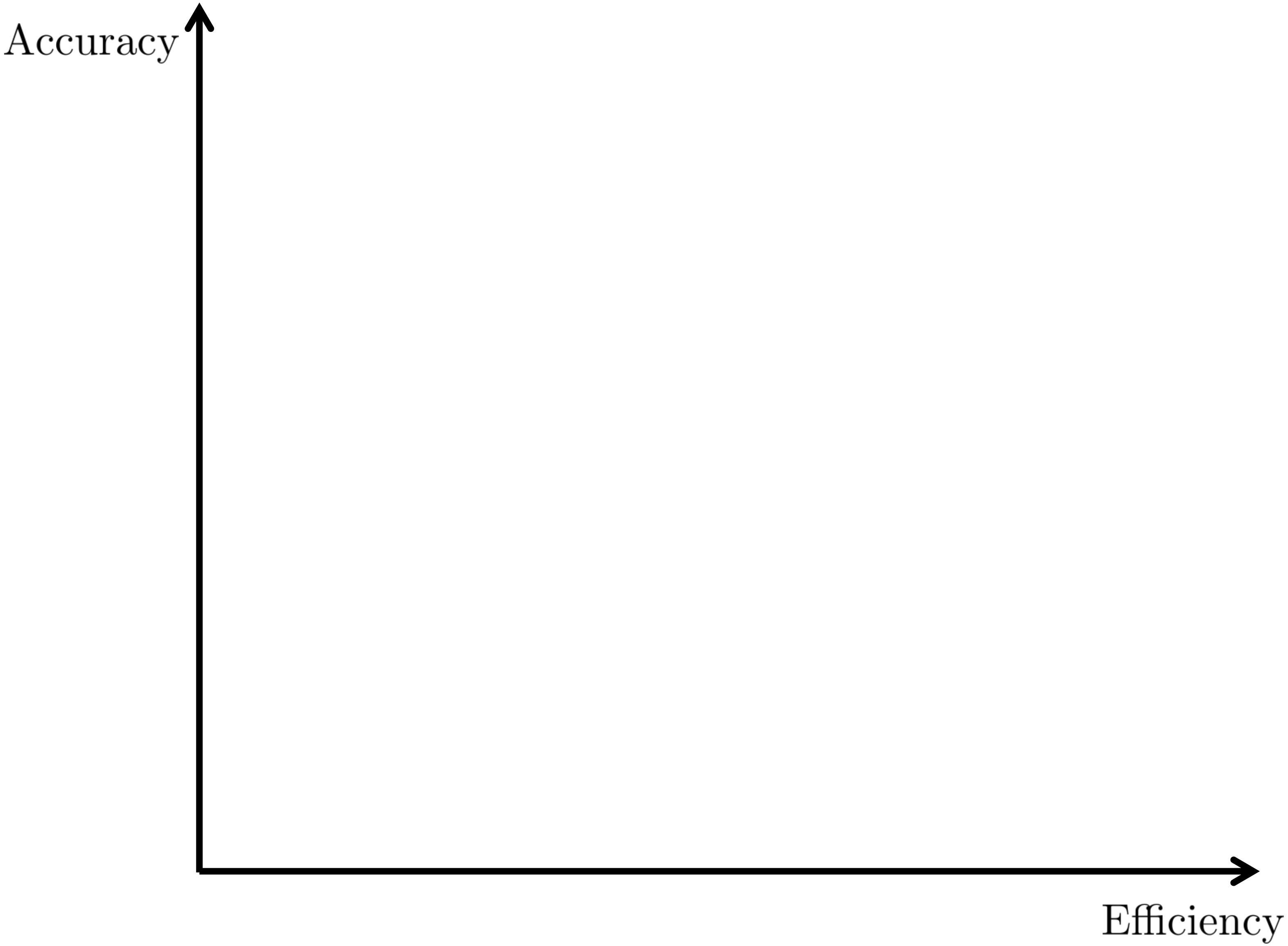
$$\frac{\partial u}{\partial t} = D\nabla^2 u + \mathcal{R}[u]$$

Mesoscale



Microscale

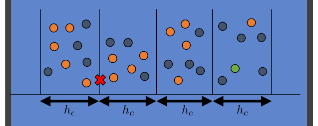




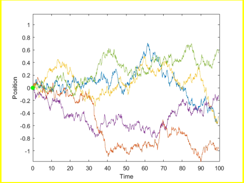
Modelling

Accuracy

Mesoscale



Microscale

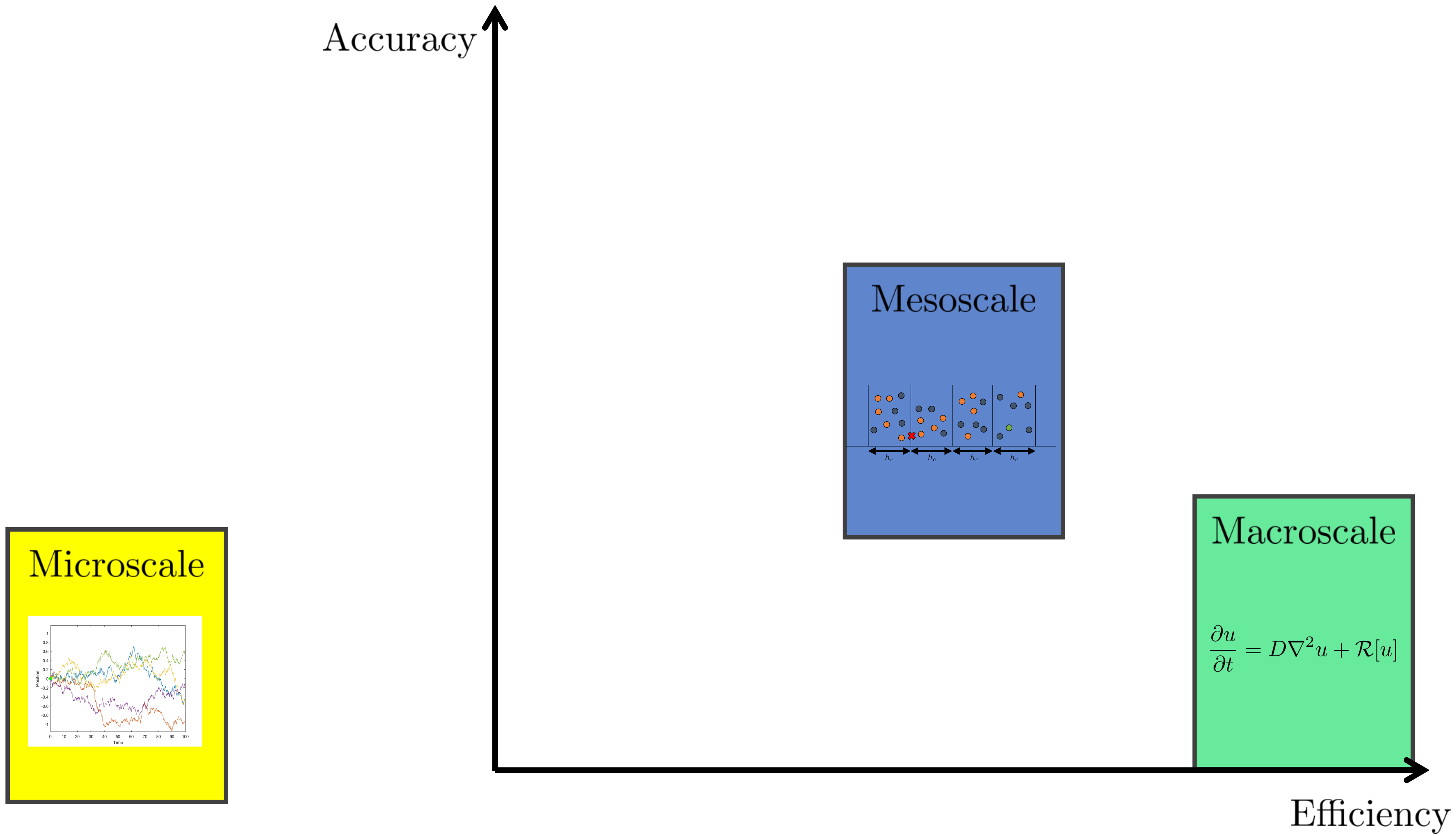


Macroscale

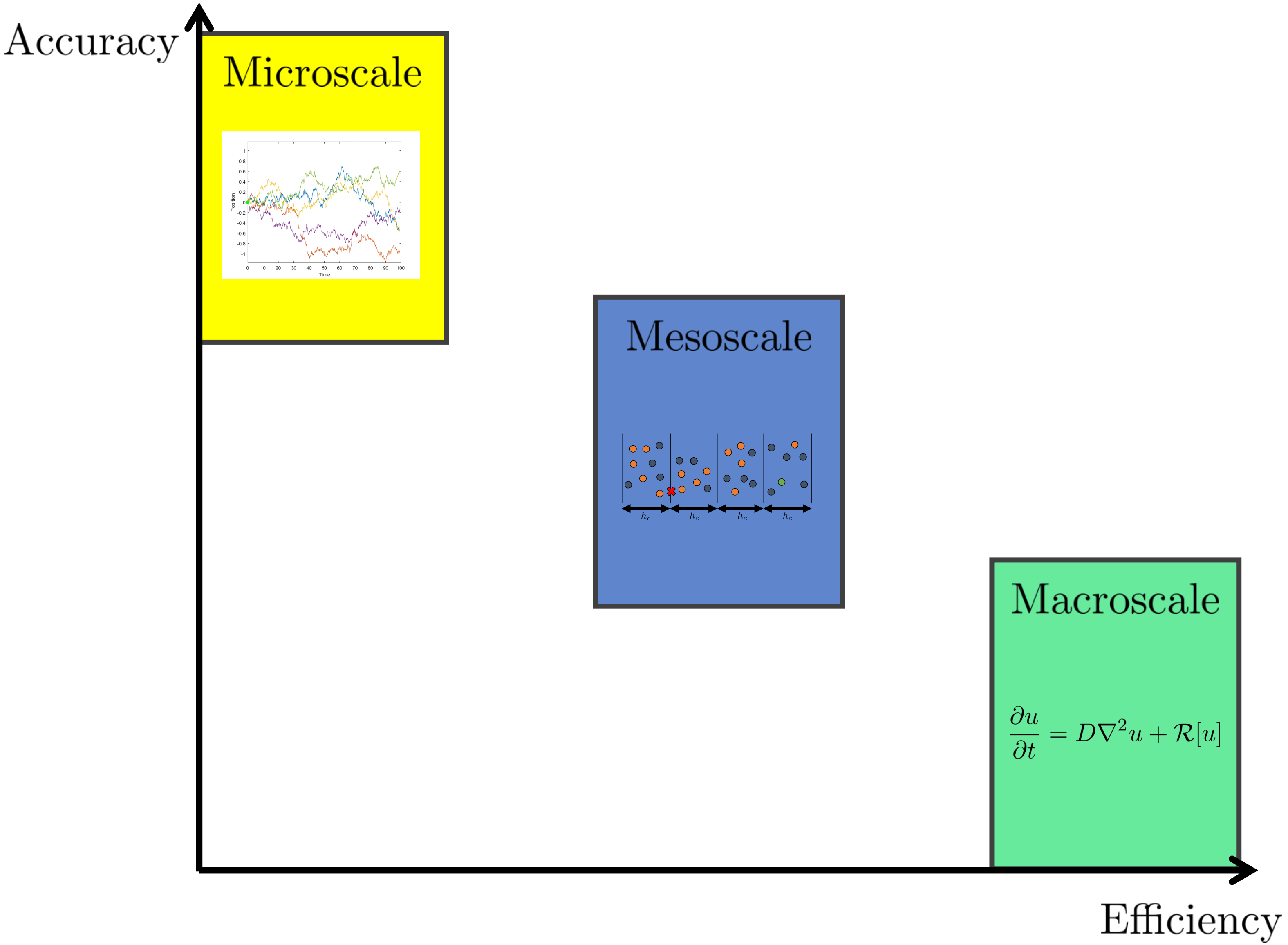
$$\frac{\partial u}{\partial t} = D\nabla^2 u + \mathcal{R}[u]$$

Efficiency

Modelling


Accuracy
Mesoscale
Microscale
Macroscale
$\frac{\partial u}{\partial t} = D\nabla^2 u + \mathcal{R}[u]$
Efficiency

Modelling


Accuracy
Microscale
Mesoscale
Macroscale
$\frac{\partial u}{\partial t} = D\nabla^2 u + \mathcal{R}[u]$
Efficiency

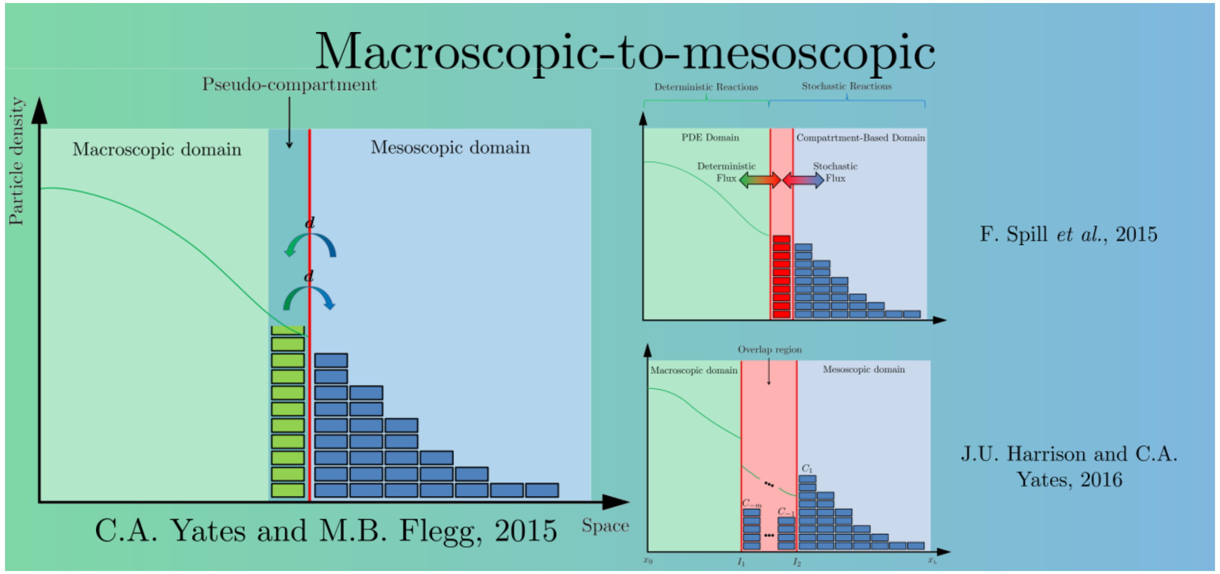
Spatially extended hybrid methods

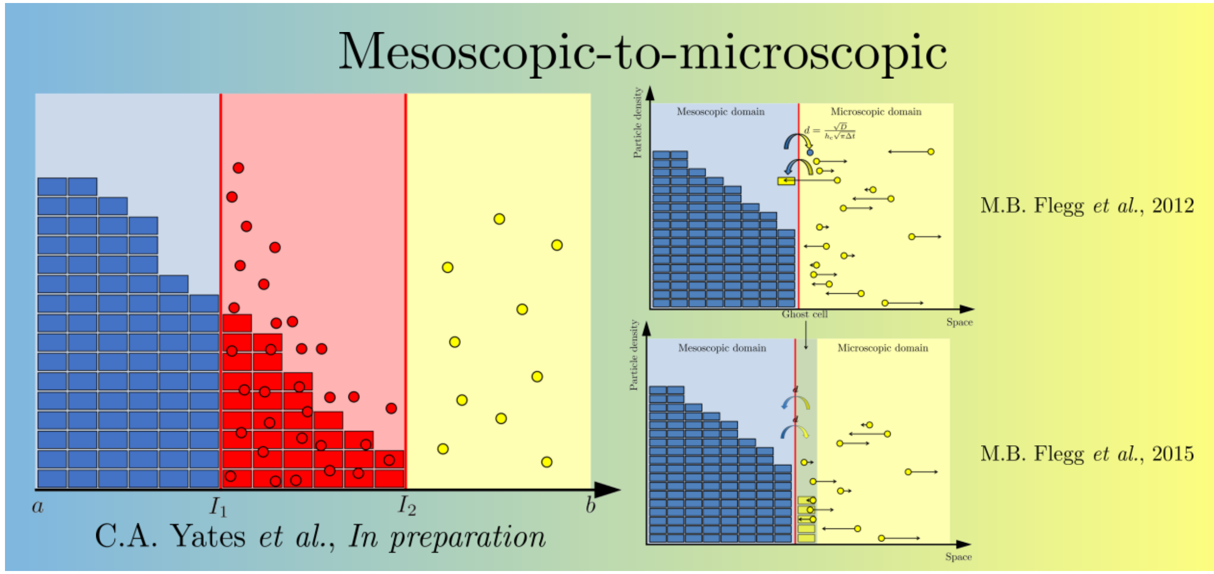
> … spatially extended hybrid methods employ different modelling paradigms at different scales in order to compliment the strengths and negate the weaknesses of each.

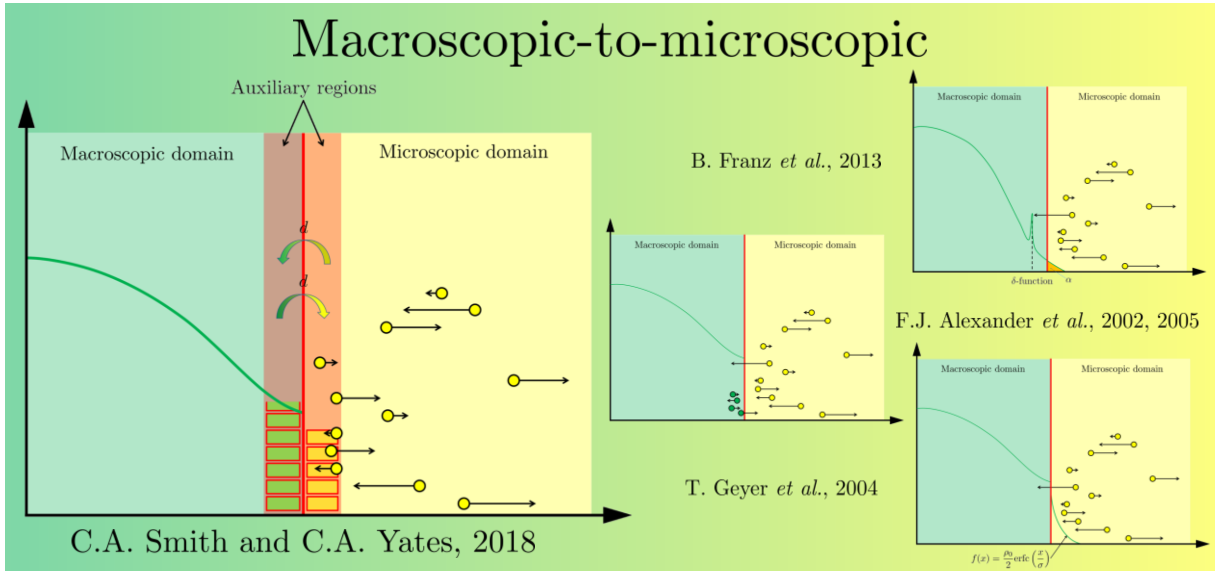
C.A. Smith and C.A. Yates, 2018

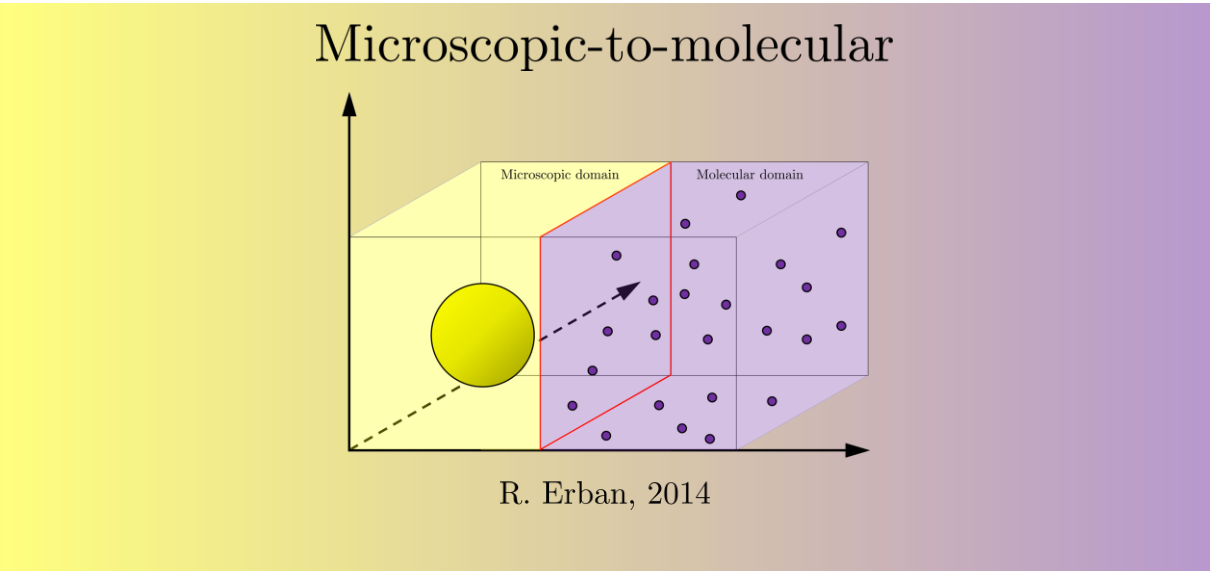
Spatially extended hybrid methods: a review

Many, many examples!



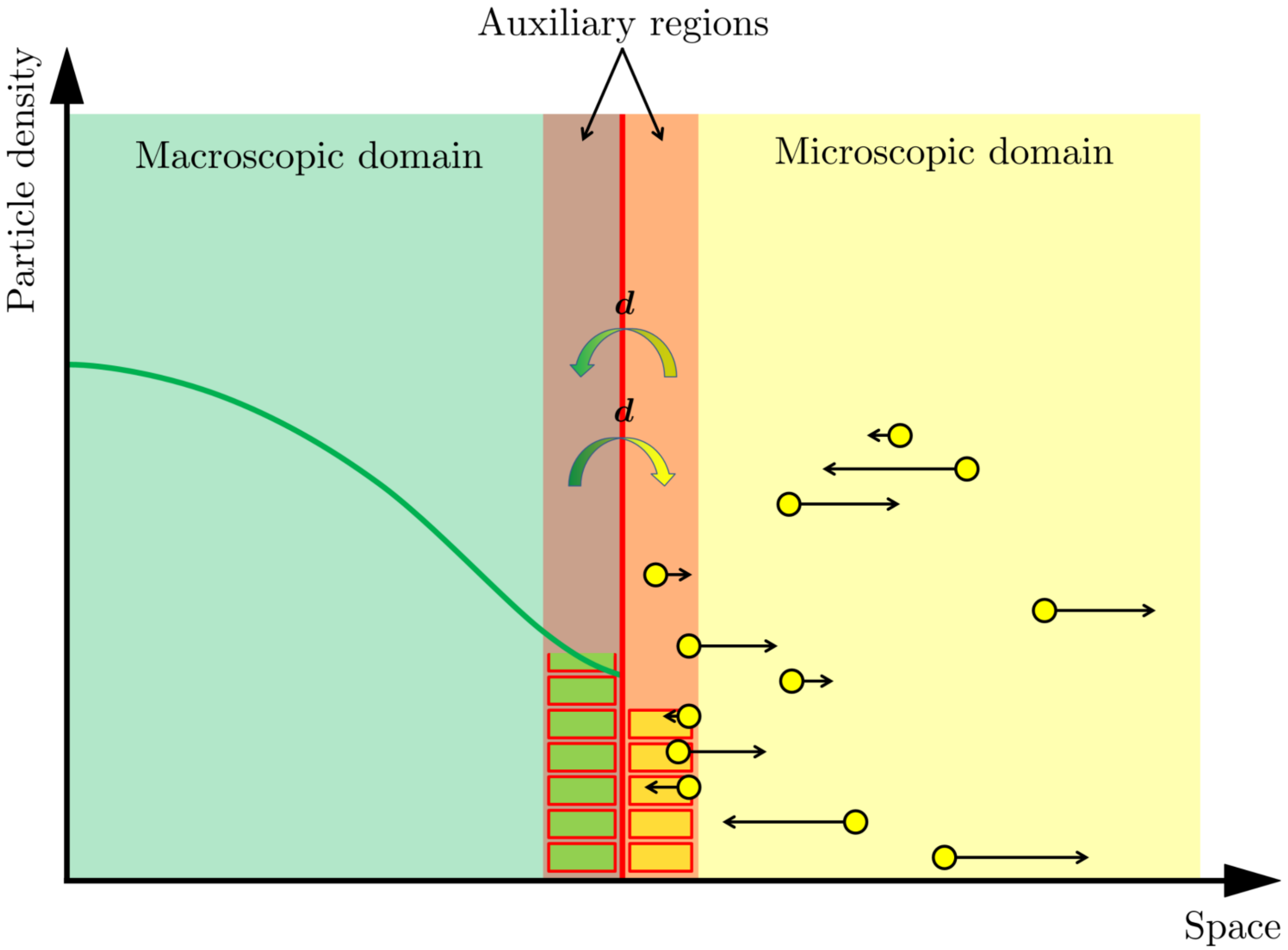






The auxiliary region method

The auxiliary region method



Basic algorithm

Let the PDE/Brownian update step be Δt, and t_Δ be the next PDE/Brownian update time.

1) Find the time until the next event within the auxiliary regions occurs. Call this t_a.
2) If this is less than the time until the next PDE/Brownian update (i.e. if $t_a < t_\Delta$), find the corresponding event and enact it.
3) Otherwise (i.e. if $t_\Delta < t_a$), evolve the PDE and Brownian subdomains.
4) Update time and return to step 1.

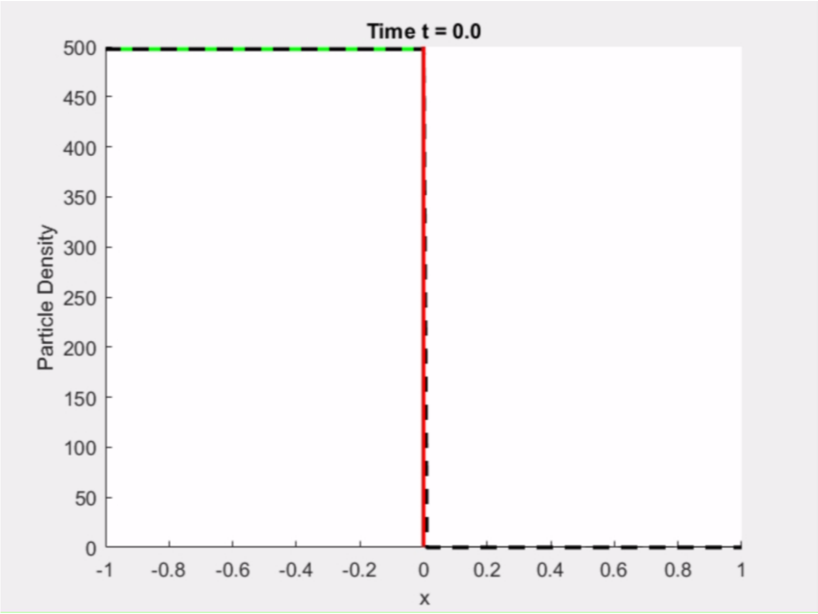
Results

Pure diffusion

$$\frac{\partial u}{\partial t} = D\nabla^2 u$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=-1} = 0 \text{ and } \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0$$

$$u(x,0) = \begin{cases} 500 & x \in [-1,0) \\ 0 & x \in [0,1] \end{cases}$$

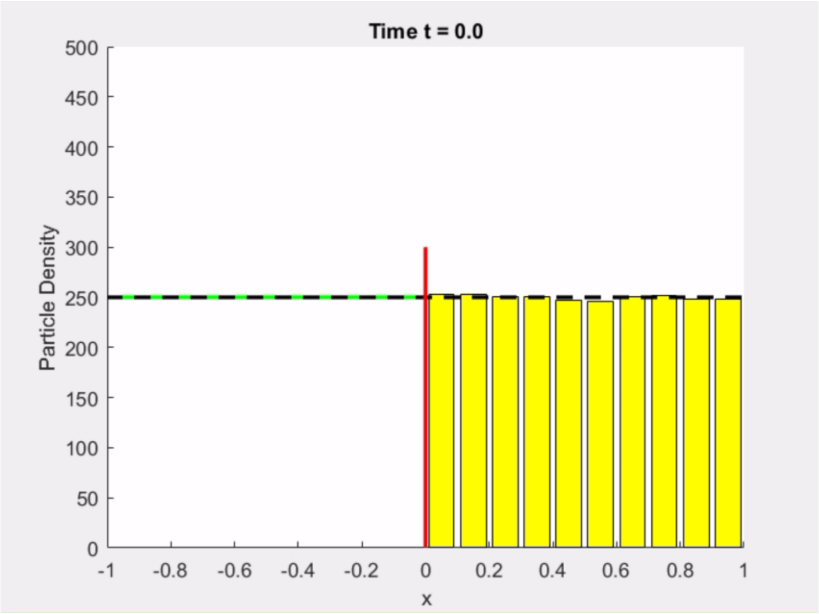


Morphogen gradient

$$\frac{\partial u}{\partial t} = D\nabla^2 u - \mu u$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=-1} = -\lambda \text{ and } \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0$$

$$u(x,0) = 250 \; \forall x \in [-1,1]$$



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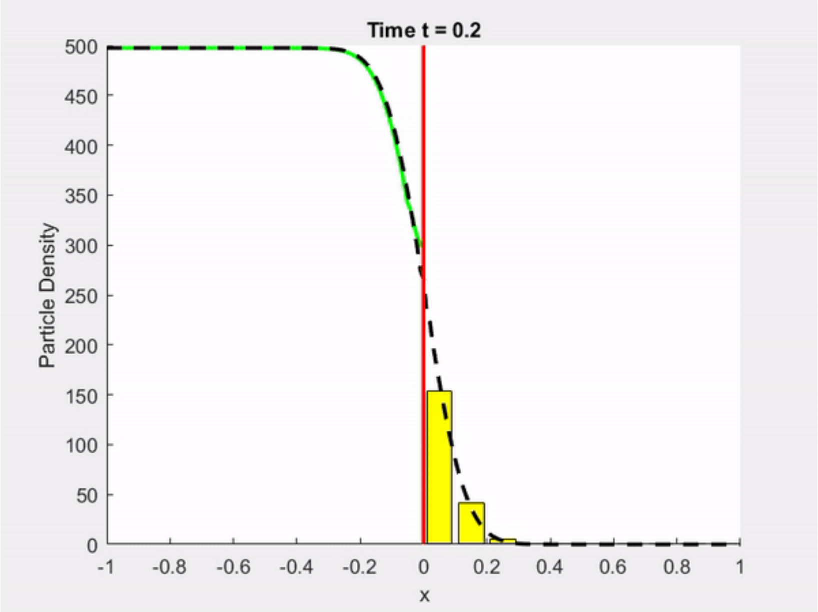

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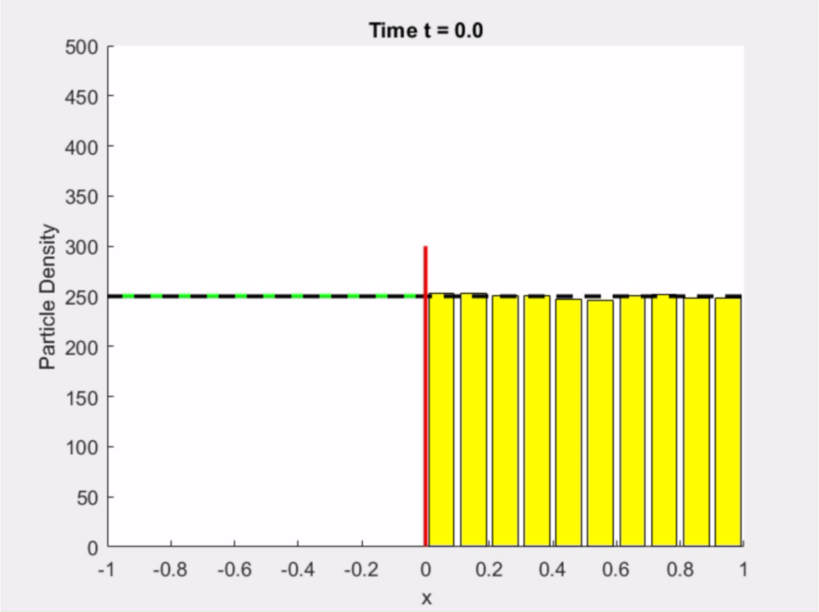

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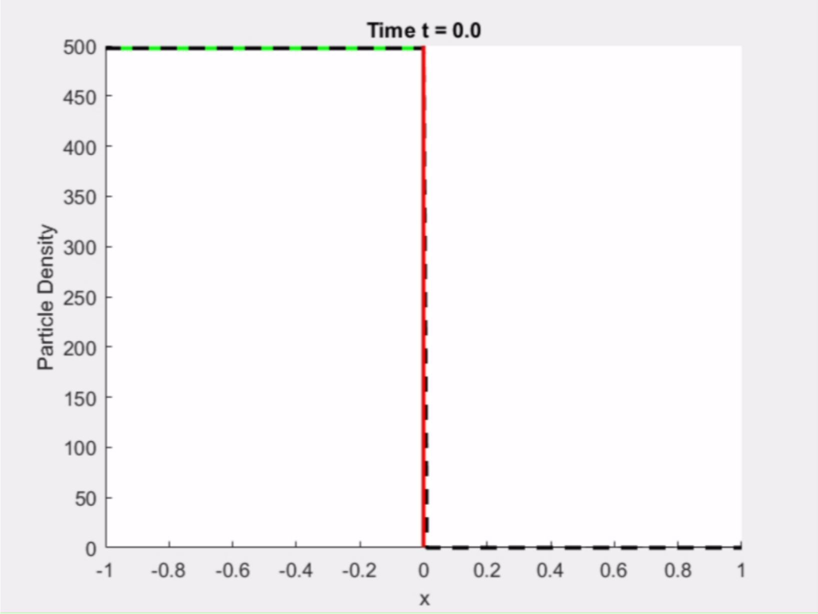

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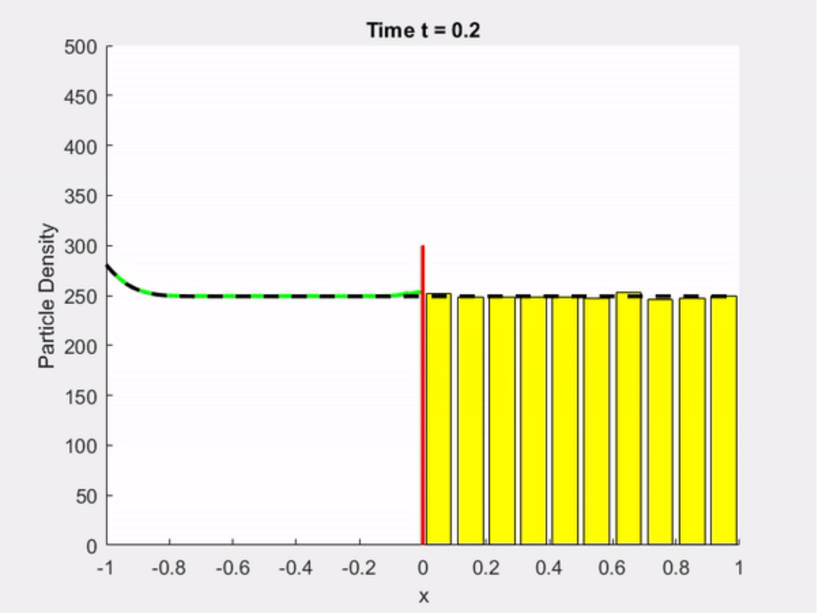

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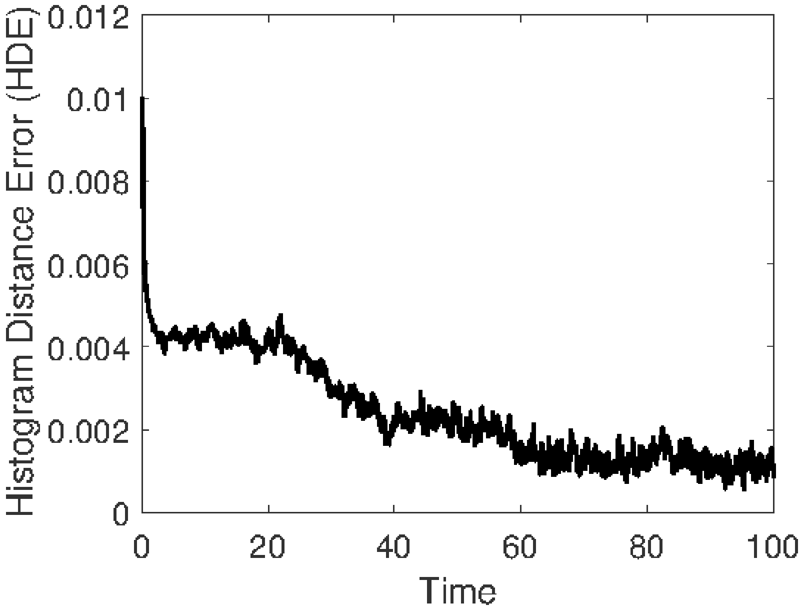

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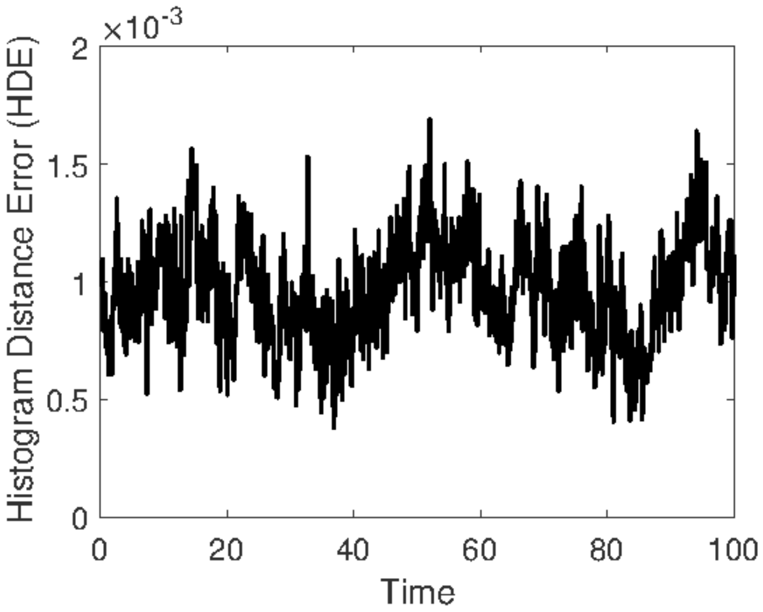
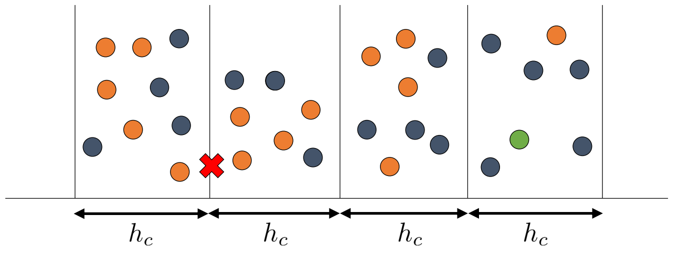
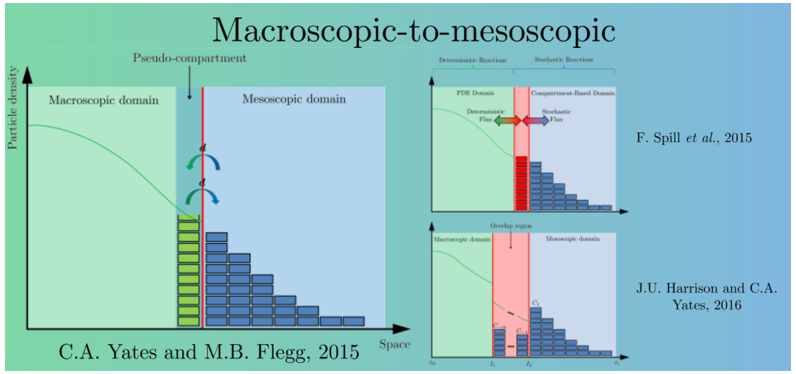


Summary

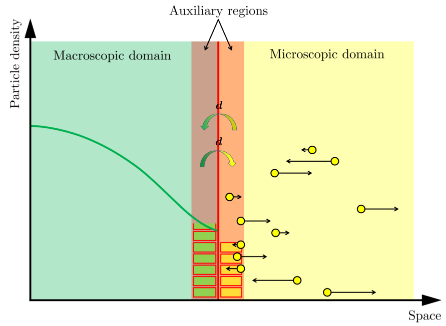
Reaction-diffusion systems may be modelled in different ways, each with complimentary advantages and disadvantages.





Hybrid methods combine these to form accurate and efficient methods. There are many examples.

The auxiliary region method combines PDE and Brownian-based approaches. Able to simulate reaction-diffusion systems accurately.



Thank you for your attention.

Any questions?

This work is joint with Kit Yates.

Get in touch:

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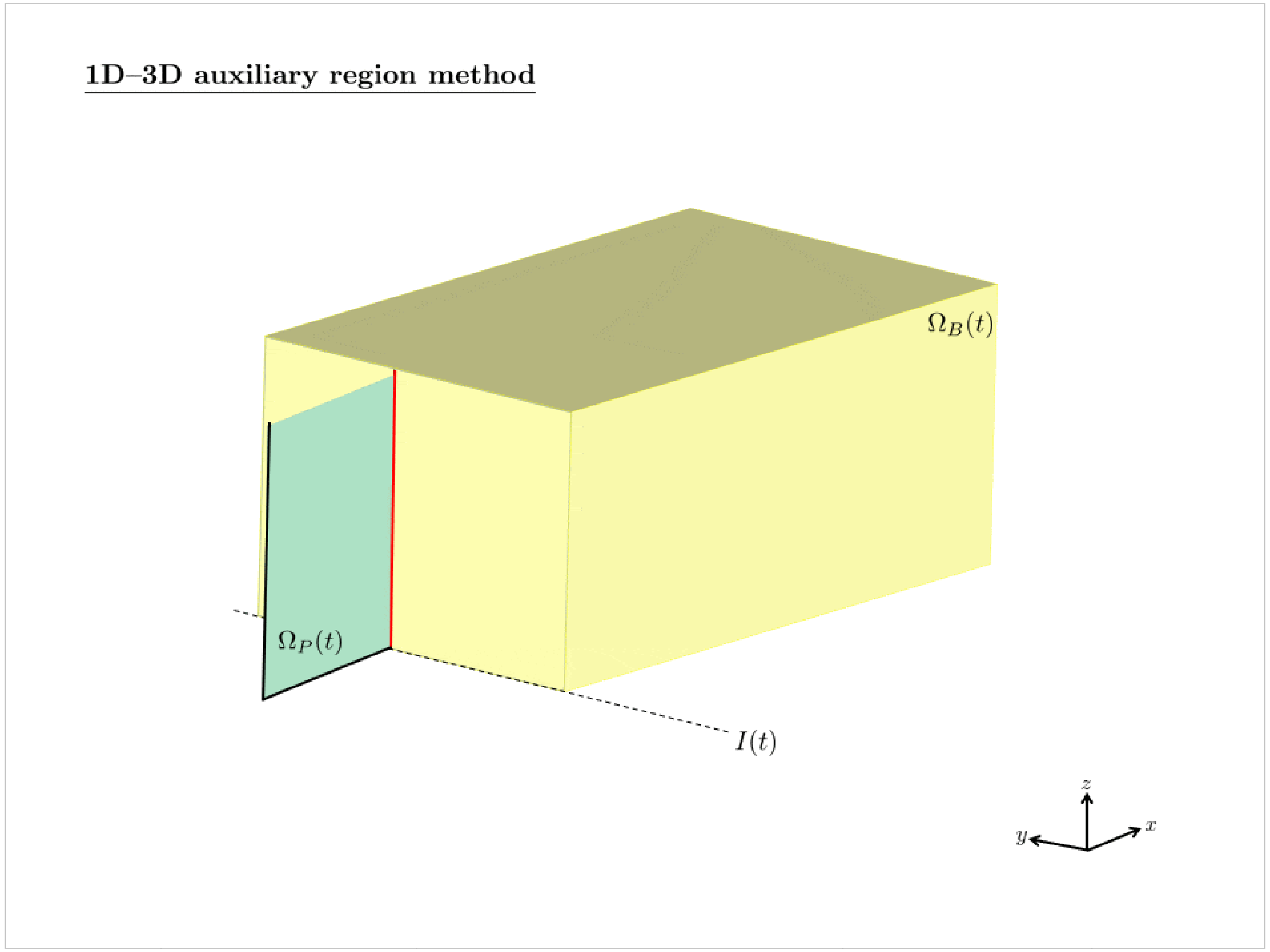
https://people.bath.ac.uk/cs640/

SAMBa

CMB Centre for Mathematical Biology University of Bath

EPSRC Engineering and Physical Sciences Research Council

1D-3D ARM

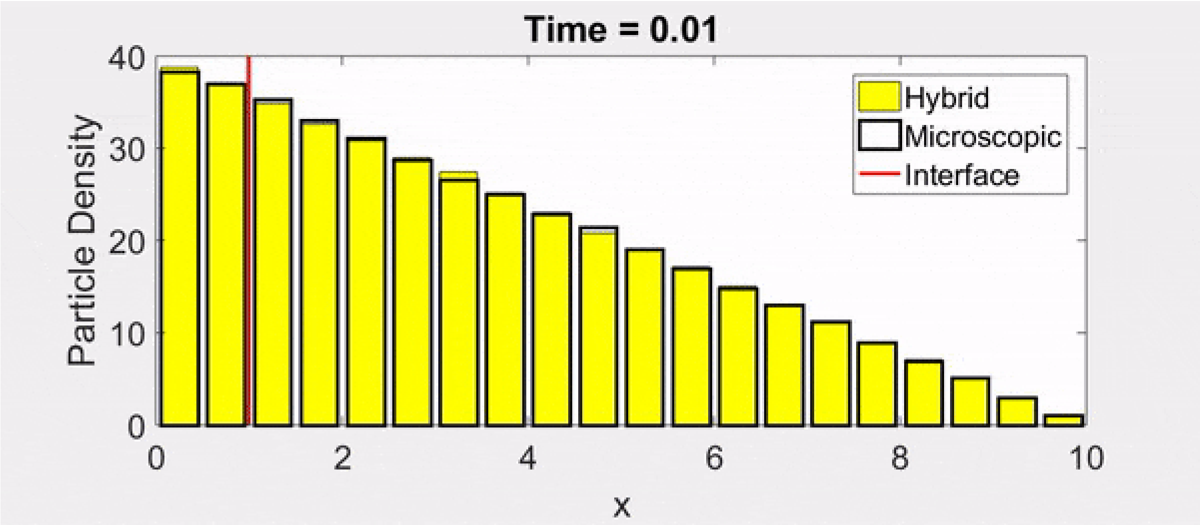


Results: 1D-3D ARM

Reaction system: $2A \xrightarrow{\kappa_1} \emptyset \xrightarrow{\kappa_2} A$

Corresponding PDE: $\dfrac{\partial u}{\partial t} = D\nabla^2 u - \dfrac{\kappa_1}{L_y L_z} u^2 + \kappa_2 L_y L_z$

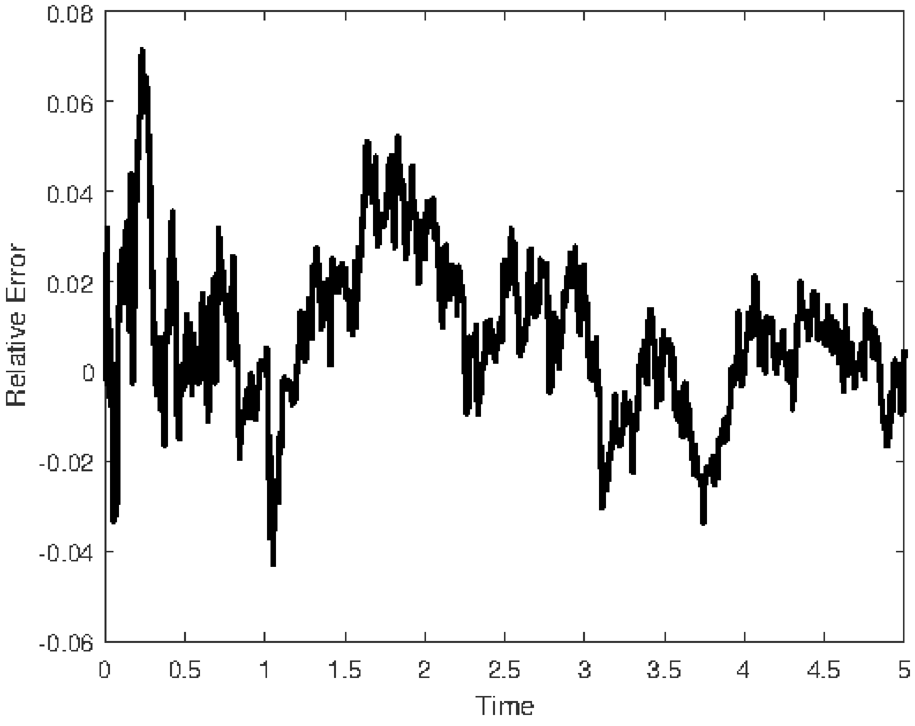
Moment closure (Poisson) $\langle A \rangle = \mathbb{V}\mathrm{ar}(A)$



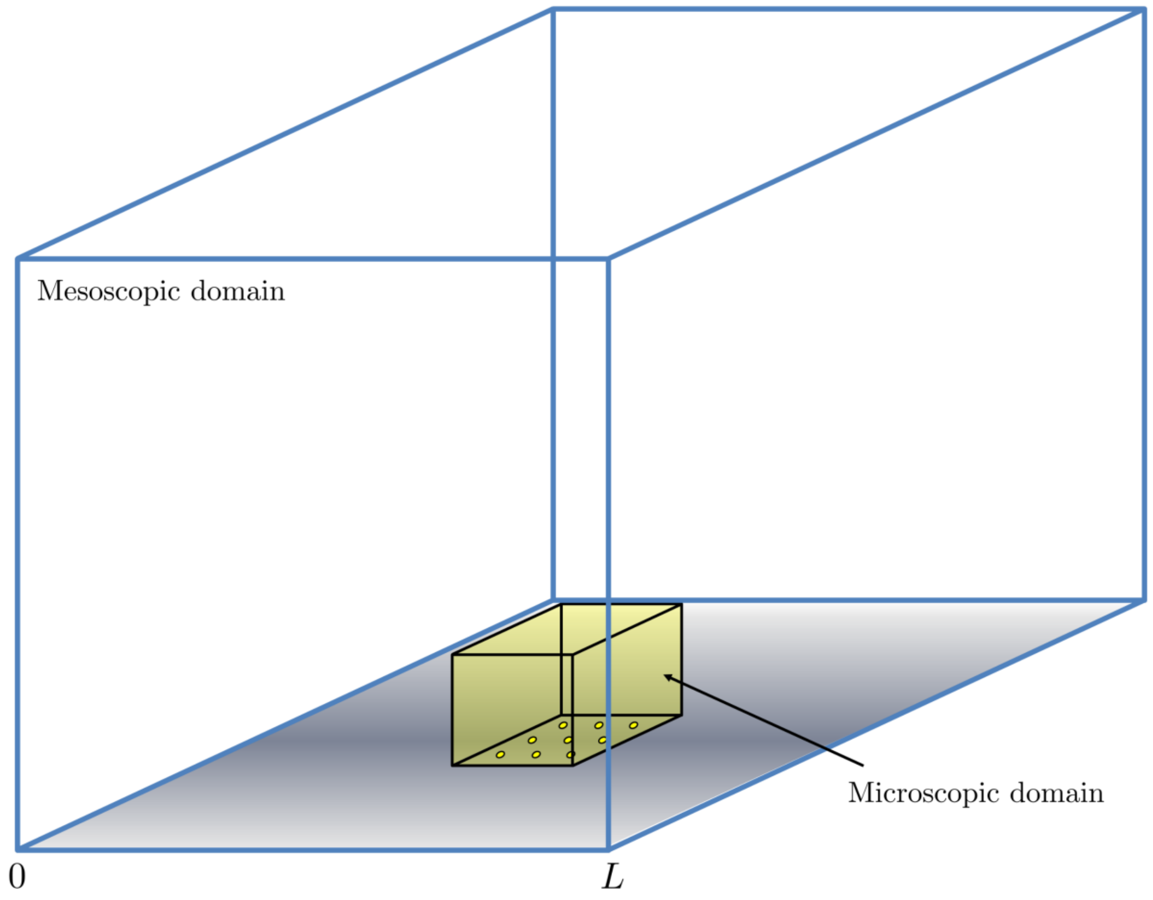
Error: 1D-3D ARM

$$E_{\mathrm{Rel}}(t) = \frac{N_{\mathrm{M}}(t) - N_{\mathrm{H}}(t)}{N_{\mathrm{M}}(t)}.$$

$N_{\mathrm{H}}(t)$ is the average number of particles in the final binning width of the hybrid method at time t, and $N_{\mathrm{M}}(t)$ is the same for the fully microscopic simulation.

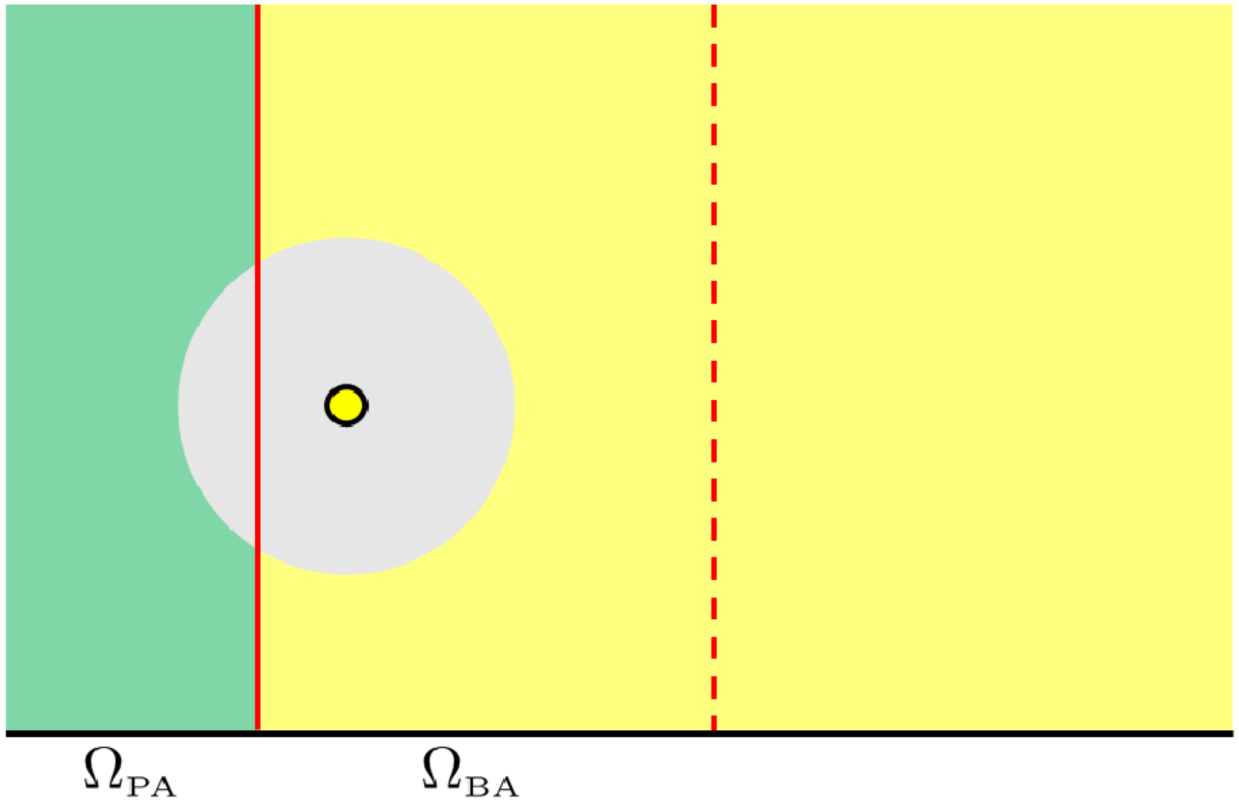


Calcium induced calcium release



Dobramysl U., Rüdiger S. and Erban R.**, Particle-based multiscale modeling of intracellular calcium dynamics**, *Multiscale. Model. Sim.*, 2015.

Reactions in the Brownian AR



Summary of models

Scale	Advantages	Disadvantages
Macroscopic (Mean-field)	Fast to compute solutions. Suitable for high copy numbers. Amenable to analysis.	Inaccurate for low particle numbers. Mean-field dynamics diverge from individual-level behaviour for high-order reactions.
Mesoscopic (Compartments)	Fast for low copy numbers. Represents the individual-level behaviour.	Can be slow for large copy numbers. Does not retain precise locations of particles or particle identity.
Microscopic (Brownian-based)	Most accurate representation of the three. Can be used for low copy numbers.	Slow to compute reactions. Impractical for large numbers of particles.